ZERO VISCOSITY LIMIT

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- Definition: $Q_T = \mathbb{R} \times (0, T)$.
- Notation: $C_b(\mathbb{R}) = C(\mathbb{R}) \cap L^\infty(\mathbb{R})$.
- Notation: $u_x = \frac{\partial u}{\partial x}, \quad u_t = \frac{\partial u}{\partial t}$.
- **THE HEAT KERNEL FOR** $u_t = \mu u_{xx}$

$$G_\mu(x,t) = \frac{1}{\sqrt{4\pi\mu t}} e^{-\frac{x^2}{4\mu t}}, \quad x \in \mathbb{R}, \ t > 0.$$

Considering the solution $u(x,t)$ satisfying, for a positive coefficient $\mu > 0$ (**viscosity coefficient**)

$$u_t + f(u)_x - \mu u_{xx} = 0, \quad x \in \mathbb{R}, \ t \geq 0, \tag{1}$$

with given $u(x,0) = u_0(x) \in C_0^\infty(\mathbb{R})$.

SOME FACTS ABOUT THE KERNEL G_μ.

Definition 1. *Let $g(x,t) \in C_b(\mathbb{R} \times [0,T])$. We say that g* ***"decays as*** *$|x| \to \infty$ if*

$$\lim_{\rho \to \infty} \max_{0 \leq \tau \leq T} |g(\pm\rho, \tau)| = 0.$$

- **Claim 2.** *Let $g(x,t) \in C_b(\mathbb{R} \times [0,T])$ decay as $|x| \to \infty$. Define*

  $$f(x,t) = \int_0^t G_\mu(x-y, t-s) g(y,s) dy\, ds.$$

  Then $f(x,t)$ decays as $|x| \to \infty$.

- **Claim 3.** *Let $k \geq 0$ be an integer. Then there exists a constant $C_k > 0$ such that*

$$\int_{\mathbb{R}} \left| \partial_x^k G_\mu(x,t) \right| dx = C_k (\mu t)^{-\frac{k}{2}}, \quad t > 0. \tag{2}$$

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Proof. Define a new variable $y = \frac{x}{\sqrt{\mu t}}$, so that $\partial_x = (\sqrt{\mu t})^{-1}\partial_y$. It is immediate that $G_\mu(x,t) = \frac{1}{\sqrt{\mu t}} G(y,1)$, hence

$$\int_{\mathbb{R}} \left|\partial_x^k G_\mu(x,t)\right| dx = (\sqrt{\mu t})^{-k}(\sqrt{\mu t})^{-1} \int_{\mathbb{R}} \left|\partial_y^k G(y,1)\right| \cdot \sqrt{\mu t}\, dy.$$

□

THE CONVECTION-DIFFUSION LINEAR EQUATION

Operator Notation:

$$\mathscr{L} = \frac{\partial}{\partial t} + a(x,t)\frac{\partial}{\partial x} - \mu\frac{\partial^2}{\partial x^2}, \tag{3}$$

where $a(x,t) \in L^\infty(Q_T)$ and $\mu > 0$.

We consider the linear equation

$$\boxed{\begin{aligned} \mathscr{L}\phi = \phi_t + a(x,t)\phi_x - \mu\phi_{xx} = 0, \quad (x,t) \in Q_T = (\mathbb{R}\times(0,T)),\ \mu > 0, \\ \phi(x,0) = \phi_0(x) \in C_0^\infty(\mathbb{R}). \end{aligned}} \tag{4}$$

Definition 4. *We say that $\phi \in C_b(\overline{Q_T})$ is a classical solution to* (4) *if*

$$\frac{\partial}{\partial t}\phi \in C_b(Q_T),\ \frac{\partial^2}{\partial x^2}\phi \in C_b(Q_T),$$

and Equation (4) *is satisfied.*

Similarly ϕ is a classical ***subsolution*** *if*

$$\mathscr{L}\phi(x,t) \le 0, \quad (x,t) \in Q_T. \tag{5}$$

MAXIMUM PRINCIPLE FOR A CLASSICAL SUBSOLUTION

We consider the inequality (5) and assume that

$$|a(x,t)| \le A, \quad (x,t) \in \overline{Q_T}. \tag{6}$$

Lemma 5. *Let $w(x,t) \in C_b(\overline{Q_T})$ be a classical subsolution of* (5) *in $Q_T = \mathbb{R}\times(0,T)$ that decays as $|x| \to \infty$. Let $w(x,0) = w_0(x) \in C_b(\mathbb{R})$ be decaying as $|x| \to \infty$. Let $N = \max_{\overline{Q_T}} w(x,t)$. Then*

$$N = \max_{\mathbb{R}} w_0(x). \tag{7}$$

Proof. Consider the function $z(x,t) = w(x,t) - (\varepsilon t + \delta x^2),\ \varepsilon,\ \delta > 0$. Fix $0 < h < T$. Assume that $z(x,t)$ attains a maximum (in $\overline{Q_{T-h}}$) at (x_0,t_0), $t_0 > 0$. Then:

$$\begin{aligned}\partial_t z(x_0,t_0) &= \partial_t w(x_0,t_0) - \varepsilon \geq 0,\\ \partial_x z(x_0,t_0) &= \partial_x w(x_0,t_0) - 2\delta x_0 = 0,\\ \partial_{xx} z(x_0,t_0) &= \partial_{xx} w(x_0,t_0) - 2\delta \leq 0,\end{aligned}$$

so that, using Equation (5) satisfied by w,

$$\begin{aligned}0 &\leq \partial_t z(x_0,t_0) - \mu\partial_{xx} z(x_0,t_0)\\ &= \partial_t w(x_0,t_0) - \varepsilon - \mu[\partial_{xx} w(x_0,t_0) - 2\delta]\\ &\leq -a(x_0,t_0)\partial_x w(x_0,t_0) - \varepsilon - 2\mu\delta\\ &\leq 2A\delta x_0 - \varepsilon - 2\mu\delta.\end{aligned}$$

Since $z(x_0,t_0) \geq z(0,0) = w(0,0) \geq \inf_{\mathbb{R}} w_0(x)$ we have $w(x_0,t_0) - \varepsilon t_0 - \delta x_0^2 \geq \inf_{\mathbb{R}} w_0(x)$ and we conclude that $\delta x_0^2 \leq b = |\inf_{\mathbb{R}} w_0(x)| + N$. In addition, $|a(x_0,t_0)| \leq A$, so finally

$$0 \leq 2A\delta|x_0| - \varepsilon - 2\mu\delta \leq 2A\sqrt{b\delta} - \varepsilon - 2\mu\delta. \tag{8}$$

Taking $\delta = \varepsilon^3$ we get a contradiction for $\varepsilon > 0$ sufficiently small.

Thus $t_0 = 0$ and

$$w(x,t) \leq (\max_{\mathbb{R}}\ w_0) + \varepsilon t + \delta x^2, \quad (x,t) \in Q_T,$$

and letting $\varepsilon, \delta \to 0$ (and then $h \to 0$) yields (7). □

We turn back to the viscous conservation law and assume that $u(x,t)$ is a classical solution of the equation.

$$\begin{aligned}u_t + f(u)_x - \mu u_{xx} = 0, \quad (x,t) \in Q_T = (\mathbb{R} \times (0,T)),\ \mu > 0,\\ u(x,0) = u_0(x) \in C_0^\infty(\mathbb{R}).\end{aligned} \tag{9}$$

Recall that we proved already the following theorem.

Theorem 6. *Assume that $u_0(x) \in C_0^\infty(\mathbb{R})$. Then the viscous conservation law* (9) *has a unique smooth solution $u(x,t)$, that decays as $|x| \to \infty$.*

Furthermore, this solution has the following properties:

- ***[Continuity in** $\overline{Q_T}$ **:]** For every integer j, $\partial_x^j u(x,t) \in C_b(\overline{Q_T})$.*
- ***[Maximum principle:]** The solution satisfies*

$$\max_{\overline{Q_T}} |u(x,t)| = \|u_0\|_{L^\infty(\mathbb{R})}.$$

Remark 7. *In fact, the uniqueness was proved for solutions satisfying*

$$\begin{aligned}u(x,t) = u^{(0)}(x,t) - \int_0^t \int_{\mathbb{R}} \partial_x G_\mu(x-y,t-s) f(u(y,s))dy\,ds,\\ (x,t) \in \overline{Q_T}.\end{aligned} \tag{10}$$

Note that (10) *implies* (9).

We now show that the classical solution $u(x,t)$ in Theorem 6 decays exponentially as $|x| \to \infty$. Let $M = \|u_0\|_{L^\infty(\mathbb{R})}$ and assume that for some $R > 0$

$$supp\ u_0 \subseteq (-R, R).$$

Lemma 8. ***[Exponential decay]*** *Let $u(x,t)$ be the classical solution in Theorem 6 and assume that for some constant $A > 0$*

$$|f'(u)| \leq A.$$

Then there exists a constant $C > 0$ such that

$$|u(x,t)| \leq M e^{Ct} e^{\frac{(A+1)t+R-|x|}{\mu}}, \quad (x,t) \in \overline{Q_T}. \tag{11}$$

Proof. Let $\mathscr{L}$ be the operator as in (3) with $a(x,t) = f'(u(x,t))$.

Let $\psi(x) \in C^\infty(\mathbb{R})$ be a function satisfying:

$$\begin{cases} |\psi'(x)| \leq 1, & x \in \mathbb{R}, \\ \psi(x) \geq 0, & |x| \leq R, \\ \psi(x) = R - |x|, & |x| > R. \end{cases} \tag{12}$$

Let $C = \max_{\mathbb{R}} |\psi''(x)|$ and define

$$\phi(x,t) = e^{Ct} e^{\frac{(A+1)t+\psi(x)}{\mu}}.$$

We compute

$$\mathscr{L}\phi = \Big[\frac{\partial}{\partial t} + a(x,t)\frac{\partial}{\partial x} - \mu\frac{\partial^2 \psi}{\partial x^2}\Big]\phi$$
$$= \Big\{C + \frac{A+1}{\mu} + a\frac{\partial_x \psi}{\mu} - \mu\Big[\frac{\partial_x^2 \psi}{\mu} + \frac{(\partial_x \psi)^2}{\mu^2}\Big]\Big\}\phi \geq 0,$$

since $|a(x,t)| \leq A$. Hence $\mathscr{L}(\pm u - M\phi) \leq 0$ and the proof is complete in view of Lemma 5. □

Remark 9. ***[Decay out of influence domain]*** *Note that if $|x| > R + (A+1)t$ then $u(x,t) \to 0$ as $\mu \to 0$.*

Next we want to see if the $x-$derivative of the solution $u(x,t)$ decays as $|x| \to \infty$. We can obtain such a result (in fact exponential decay) as follows.

Lemma 10. ***[Exponential decay of derivative]*** *Let $u(x,t)$ be the classical solution as in Lemma 8 Then the derivative $u_x(x,t)$ decays exponentially as $|x| \to \infty$, $t \in [0,T]$.*

Proof. We use the notation introduced in Lemma 8 and its proof.

Recall the Landau-Kolmogorov inequality: If $g(x) \in C^2[-1,1]$ then there exists a constant $K > 0$ such that

$$|g'(0)| \leq K[\|g\|_{L^\infty[-1,1]} + \|g\|^{\frac{1}{2}}_{L^\infty[-1,1]} \cdot \|g''\|^{\frac{1}{2}}_{L^\infty[-1,1]}].$$

Take any $x > 2[(A+1)T + R + 1]$ and apply the inequality to the solution $g(x) = u(x,t)$ in the interval $[x-1, x+1]$. In view of (11)

$$|u(x,t)| \leq M e^{Ct} e^{\frac{(A+1)t+R-|x|}{\mu}}, \quad (x,t) \in Q_T,$$

and by Theorem 6 there exists a constant $B > 0$ such that

$$|u_{xx}(x,t)| \leq B, \quad (x,t) \in Q_T.$$

Thus, the Landau-Kolmogorov inequality yields

$$|u_x(x,t)| \leq K\Big[Me^{CT}e^{\frac{-\frac{1}{2}x}{\mu}} + B^{\frac{1}{2}} \cdot M^{\frac{1}{2}}e^{\frac{1}{2}CT}e^{\frac{-\frac{1}{4}x}{\mu}}\Big], \quad x > 2[(A+1)T + R + 1].$$

An analogous estimate holds for $x \to -\infty$, and this concludes the proof of the lemma. $\square$

THE L^1 CONTRACTION PROPERTY

We assume that $u(x,t)$ is a classical solution of (1) in $Q_T = \mathbb{R} \times (0,T)$ and let $v(x,t) \in C_b(\overline{Q_T})$ be another classical solution

$$v_t + f(v)_x - \mu v_{xx} = 0, \quad x \in \mathbb{R},\ (x,t) \in Q_T, \tag{13}$$

subject to $v(x,0) = v_0(x) \in C_0^\infty(\mathbb{R})$.

Denote $w = u - v$. It satisfies

$$w_t + (f(u) - f(v))_x - \mu w_{xx} = 0.$$

We have

$$f(u) - f(v) = \int_0^1 \frac{d}{d\lambda} f(\lambda u + (1-\lambda)v)d\lambda = \int_0^1 f'(\lambda u + (1-\lambda)v)d\lambda \cdot (u - v).$$

Let $a(x,t) = \int_0^1 f'(\lambda u + (1-\lambda)v)d\lambda$. In view of the maximum principle (satisfied by u, v) we get

$$|a(x,t)| \leq A = \max\left\{|f'(\xi)|,\ |\xi| \leq \max(\|u_0\|_{L^\infty(\mathbb{R})}, \|v_0\|_{L^\infty(\mathbb{R})})\right\}.$$

We conclude that the function $w(x,t)$ satisfies the convection-diffusion equation

$$\begin{aligned} w_t + (a(x,t) \cdot w)_x - \mu w_{xx} = 0, \quad (x,t) \in Q_T, \\ w(x,0) = w_0(x) = u_0(x) - v_0(x). \end{aligned} \tag{14}$$

Let $z(x,t)$ be the "dual" equation in Q_T, namely

$$\begin{aligned} z_t + a(x,t) \cdot z_x + \mu z_{xx} = 0, \\ z(x,T) = z_T(x) \in C_0^\infty(\mathbb{R}). \end{aligned} \tag{15}$$

Note that this is a "backward" heat equation. Changing to $\tau = T - t$ we get

$$\begin{aligned} z_\tau - a(x,t) \cdot z_x - \mu z_{xx} = 0, \\ z(x,0) = z_T(x). \end{aligned}$$

In view of Lemma 5 the function $z(x,t)$ satisfies the maximum principle and in view of Lemma 8 the function $w(x,t)$ decays exponentially as $|x| \to \infty$.

We now have all the tools to prove the following theorem.

Theorem 11. *Fix (any) $T > 0$. Let $u(x,t),\ v(x,t) \in C_b(\overline{Q_T})$ be classical solutions to the viscous conservation laws of (1) and (13), respectively, in $Q_T = \mathbb{R} \times (0,T)$. Then*

$$\int_{\mathbb{R}} |u(x,T) - v(x,T)|dx \le \int_{\mathbb{R}} |u_0(x) - v_0(x)|dx. \tag{16}$$

*In other words, the "evolution map" $u_0 \to u(\cdot,t)$ is a **contraction** in $L^1(\mathbb{R})$.*

Proof. Multiplying Equation (14) by $z(x,t)$, integrating by parts and noting that there are no boundary terms as $|x| \to \infty$ (by exponential decay), and using the equation (15) satisfied by z, we obtain

$$\int_{\mathbb{R}} w(x,T)z_T(x)dx = \int_{\mathbb{R}} w_0(x)z(x,0)dx. \tag{17}$$

Let us take $z_T(x) \in C_0^\infty(\mathbb{R})$ such that $\|z_T\|_{L^\infty(\mathbb{R})} \le 1$. By the maximum principle (applied to (15)) it follows that $\|z(\cdot,0)\|_{L^\infty(\mathbb{R})} \le 1$. From Equation (17) we have

$$\Big|\int_{\mathbb{R}} w(x,T)z_T(x)dx\Big| \le \int_{\mathbb{R}} |w_0(x)|dx.$$

Taking the supremum over the set $\left\{z_T(x) \in C_0^\infty(\mathbb{R}),\ \|z_T\|_{L^\infty(\mathbb{R})} \le 1\right\}$ we obtain (16). $\square$

Remark 12. *Note in particular that taking $v_0 = 0$ in (16) we get for any $t > 0$,*

$$\int_{\mathbb{R}} |u(x,t)|dx \le \int_{\mathbb{R}} |u_0(x)|dx. \tag{18}$$

Corollary 13. ***[uniqueness]*** *Let $u(x,t),\ v(x,t) \in C_b(\overline{Q_T})$ be classical solutions to the viscous conservation laws of (1) and (13), respectively, in $Q_T = \mathbb{R} \times (0,T)$. Suppose that $u(x,0) = v(x,0)$. Then $u(x,t) \equiv v(x,t)$ in Q_T.*

TOTAL VARIATION ESTIMATES

Let $u(x,t)$ be a classical solution to (1). Then (Galilean invariance), given $h > 0$, the function $u(x+h,t)$ is the (by uniqueness) classical solution, subject to the initial function $u_0(x+h)$. The contraction property (16) yields

$$\int_{\mathbb{R}} |u(x+h,t) - u(x,t)|dx \le \int_{\mathbb{R}} |u_0(x+h) - u_0(x)|dx. \tag{19}$$

But $u_0(x+h) - u_0(x) = \int_0^1 \frac{d}{d\lambda} u_0(x+\lambda h)d\lambda = h\int_0^1 u_0'(x+\lambda h)d\lambda$, hence

$$\int_{\mathbb{R}} |u_0(x+h) - u_0(x)|dx \le h\int_{\mathbb{R}} |u_0'(x)|dx = hTV(u_0). \tag{20}$$

Let $K > 0$. From (19) and (20) we conclude

$$\int_{-K}^{K} \Big|\frac{u(x+h,t) - u(x,t)}{h}\Big|dx \le TV(u_0(x)).$$

Taking the limit $h \to 0$ and using the dominated convergence theorem we have

$$\int_{-K}^{K} |\partial_x u(x,t)|dx \le TV(u_0(x)).$$

Sending $K \to \infty$ we finally have

Theorem 14. *The total variation of the solution $u(x,t)$, for any fixed $t > 0$, is bounded by $TV(u_0)$. In particular, this bound is independent of the viscosity coefficient μ.*

ZERO VISCOSITY LIMIT

We note that the maximum principle and the total variation estimates **do not depend on the viscosity coefficient** μ. Since now this coefficient plays a crucial role, we refer to it explicitly and denote the solution to (1) as $u_\mu(x,t)$.

Let $\{t_j\}_{j=1}^\infty \subseteq [0,T]$ be a dense sequence (e.g., all rationals). By **Helly's theorem** we have the following lemma.

Lemma 15. *There exists a decreasing sequence $\{\mu_k \downarrow 0\}_{k=1}^\infty$ and limiting functions $\{u(x,t_j)\}_{j=1}^\infty$ such that*

$$\lim_{k\to\infty} u_{\mu_k}(x,t_j) = u(x,t_j), \quad x \in \mathbb{R},\ j = 1,2,\dots. \tag{21}$$

Recall that for every $\mu > 0$ the function $u_\mu(x,t)$ decays exponentially as $|x| \to \infty$, uniformly in $t \in [0,T]$. We need to show that the map

$$t \to u_\mu(\cdot,t) \in L^1(\mathbb{R})$$

is continuous, with modulus of continuity that is **independent of** $\mu > 0$. This is stated in the following lemma.

Lemma 16. *There exist constants $C, \varepsilon_0 > 0$ such that for $0 \le t < t+h \le T$,*

$$\int_{\mathbb{R}} |u_\mu(x,t+h) - u_\mu(x,t)|dx \le C\Big[\varepsilon TV(u_0) + h(\frac{1}{\varepsilon} + \frac{\mu}{\varepsilon^2})\int_{\mathbb{R}} |u_0(x)|dx\Big], \quad 0 < \varepsilon < \varepsilon_0. \tag{22}$$

Proof. For simplicity

$$w(x) = u_\mu(x,t+h) - u_\mu(x,t).$$

If $\phi(x) \in C^\infty(\mathbb{R})$ with bounded derivatives then

$$\begin{aligned} \int_{\mathbb{R}} w(x)\phi(x)dx &= \int_{\mathbb{R}}\int_t^{t+h} \frac{\partial_s u_\mu(x,s)}{\partial s}\phi(x)ds\,dx \\ &= \int_{\mathbb{R}}\int_t^{t+h} f(u_\mu(x,s))\phi'(x)ds\,dx + \mu\int_{\mathbb{R}}\int_t^{t+h} u_\mu(x,s)\phi''(x)ds\,dx. \end{aligned} \tag{23}$$

Note that the integration by parts in $\int_{\mathbb{R}} f(u_\mu(x,s))_x\phi(x)\,dx$ and $\int_{\mathbb{R}} \partial_x^2 u_\mu(x,s)\phi(x)\,dx$ are possible since $u_\mu(x,t)$ and $\partial_x u_\mu(x,t)$ decay at infinity (see Lemma 10, for which we also need $f'(0) = 0$).

Let $0 \le \chi(x) \in C_0^\infty(-1,1)$ with $\int_{-1}^1 \chi(x)dx = 1$ and define

$$\phi(x) = \int_{\mathbb{R}} sgn\, w(x-\varepsilon y)\chi(y)dy = \frac{1}{\varepsilon}\int_{\mathbb{R}} sgn\, w(z)\chi(\frac{x-z}{\varepsilon})dz.$$

Then

$$|\phi'(x)| \le \frac{C}{\varepsilon}, \quad |\phi''(x)| \le \frac{C}{\varepsilon^2}.$$

Now
(24)

$$\int_{\mathbb{R}} w(x)\phi(x)dx - \int_{\mathbb{R}} |w(x)|dx = \int_{\mathbb{R}}\int_{-1}^{1} w(x)sgn\, w(x-\varepsilon y)\chi(y)dy\, dx$$

$$- \int_{\mathbb{R}}\int_{-1}^{1} w(x)sgn\, w(x)\chi(y)dy\, dx = \int_{\mathbb{R}}\int_{-1}^{1}(w(x)-w(x-\varepsilon y))sgn\, w(x-\varepsilon y)\chi(y)dy\, dx.$$

Note

$$\int_{\mathbb{R}} |w(x)-w(x-\varepsilon y)|dx \le 2\varepsilon TV(u_0).$$

Combining (23) and (24) (and noting (18)) we obtain (22). □

From Lemma 15 and Lemma 16 we get the following theorem.

Theorem 17. *There exists a decreasing sequence $\{\mu_k \downarrow 0\}_{k=1}^{\infty}$ and a limiting function $u(x,t)$, $(x,t) \in Q_T$, such that, for every $t \in (0,T)$ and every $K > 0$,*

$$\lim_{k\to\infty}\int_{-K}^{K} |u_{\mu_k}(x,t) - u(x,t)|dx = 0. \tag{25}$$

The function $u(x,t)$ is a weak solution to

$$u_t(x,t) + f(u(x,t))_x = 0, \quad (x,t) \in Q_T. \tag{26}$$

Proof. The solution $u_\mu(x,t)$ satisfies the equation (9), namely,

$$\partial_t u_\mu + \partial_x f(u_\mu) - \mu\partial_x^2 u_\mu = 0, \quad (x,t) \in Q_T = (\mathbb{R}\times(0,T)),\ \mu > 0,$$
$$u(x,0) = u_0(x) \in C_0^\infty(\mathbb{R}).$$

Let $\phi(x,t) \in C_0^\infty(Q_T)$ be a test function. Multiplying the equation by $\phi(x,t)$ and integrating by parts, we get
(27)

$$\int_0^T\int_{\mathbb{R}}\Big[-u_\mu(x,t)\partial_t\phi(x,t) - f(u_\mu(x,t))\partial_x\phi(x,t) - \mu u_\mu(x,t)\partial_x^2\phi(x,t)\Big]dx\, dt = 0.$$

We now take $\mu = \mu_k$ (as in (25)) and let $k \to \infty$. □

References

[1] L. C. Evans, *"Partial Differential Equations"*, American Mathematical Society 1998.
[2] E. Godlewski and P.-A. Raviart, *"Hyperbolic Systems of Conservation Laws"*, Ellipses, 1991.

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